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OPTIMAL BEAM FREQUENCIES BY THE FINITE ELEMENT DISPLACEMENT METHOD

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Abstract—A finite element displacement formulation is used to maximize the first mode natural frequency of a vibrating beam of specified volume with elastically restrained ends and resting on a continuous elastic foundation subject to a constraint of minimum allowable moment of inertia. For cross-sections with moment of inertia and area related by $I = \rho A^n$ (ρ and n are positive constants), the optimality condition is reduced to a relation between the strain energy and kinetic energy densities. Beginning with a uniform beam, an iterative procedure is used to converge to the optimum material distribution and maximum first mode frequency. Results are presented for various boundary conditions, with and without the effect of any given arbitrarily varying axial load distribution and/or dead mass distribution for $n = 1$, 2 and 3.

NOTATION

$A(x)$	cross-sectional area distribution
c	constant of optimality
c^{r+1}	constant in the recurrence relation
E	Young's Modulus of Elasticity
g	acceleration due to gravity
$I(x)$	cross-sectional moment of inertia distribution
k_T^0	translational spring stiffness (at $x = 0$)
k_R^0	rotational spring stiffness (at $x = 0$)
k_T^L	translational spring stiffness (at $x = L$)
k_R^L	rotational spring stiffness (at $x = L$)
$[k_i]$	stiffness matrix of the ith element
$[K]$	assembled nonsingular stiffness matrix of the entire beam including the effect of the elastic foundation, elastic restraints and the given applied axial loading
L	total length of the beam
$[m_i]$	mass matrix of the ith element
$m(x)$	mass distribution
$m_d(x)$	dead mass distribution
m_{ci}	concentrated dead mass at the point i
$[M]$	assembled mass matrix for the entire beam
n	index in the moment of inertia-area relation
p	exponent in the recurrence relation
P_0	axial load at $x = 0$
P_L	axial load at $x = L$
$\{q\}$	vector of the independent degrees of freedom
$\{q_i\}$	displacement vector of the ith element
$q(x)$	internal axial load distribution
$r, r+1$	superscripts used to denote the (r)th and $(r+1)$st iterations
$S(x)$	applied axial load distribution
U_{bi}	bending energy of the ith element
U_B	total bending energy of the entire beam
U_{ti}	kinetic energy of the ith element

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U_T	total kinetic energy of the entire beam
v_i	volume of the ith element
V	total volume of the beam
$w(x)$	mode shape; eigenfunction
β	foundation modulus in force/(length)^2
γ	specific weight of the material of the beam
ρ	shape constant
ω_1^2	square of the first mode frequency; eigenvalue

INTRODUCTION

THE interest in minimum weight design of columns and beams dates back to around 1770 when Lagrange first tackled the problem of finding the optimal shape of the Euler–Bernoulli column but arrived at the wrong result due to computational error. The correct solution was subsequently given by Clausen in 1851 and independently by Keller [1] in 1960. Later Keller and Niordson [2] also treated the problem of the tallest column. The problem of the strongest column was generalized to all classical boundary conditions for columns with similar cross-sections, i.e. $n = 2$ by Tadjbakhsh and Keller [3]. Prager and Taylor [4] gave exact solution for a simply-supported column of sandwich construction. These exact solutions indicate that the stiffness must vanish at some points along the length of the column depending upon the boundary conditions. This undesirable feature was later removed by the present authors [5] by the use of the inequality constraint and the problem was further generalized to all possible boundary conditions, i.e. with elastic restraints, using the finite element displacement method to obtain approximate numerical solutions.

The problem of the design of a vibrating string with variable density for a specified period and the type of vibration was first considered by Rayleigh [6]. Subsequently Bessack [7], Schwarz [8–10] investigated the effect of density variation on the extreme values of the natural frequencies of strings, beams and plates. However, the most significant contributions to the present problem would be those of Niordson [11], Turner [12], Taylor [13, 14], Brach [15], and Karihaloo and Niordson [16]. Niordson treated the problem of the optimal design of a simply-supported vibrating beam through variational formulation. Turner obtained exact and finite element solutions of minimum mass design, for a specified frequency, of bars and beams fastened at one end with a mass attached at the other end. Taylor also obtained solutions, through the variational formulation, for the axial vibrations of optimum bars with and without the inequality constraint and also for the transverse vibrations of an optimum cantilever sandwich beam with a distributed mass loading. Brach on the other hand obtained exact solutions for the extremal frequencies of transverse vibrations of beams for all classical boundary conditions and for a relation between the moment of inertia and area of the form $I(x) = c_0 + \rho A(x)$. Finally, Karihaloo and Niordson provided quasi-exact solutions to the problem of optimum vibrating cantilevers [$I(x) = \rho A^n(x)$, $n = 1, 2, 3$] with or without a concentrated dead mass at the tip. The present work is an application of the finite element displacement method, similar to that of Ref. [5], to the optimal design of vibrating beams on a continuous elastic foundation with all possible boundary conditions, i.e. with elastic restraints. In addition the inequality constraint of minimum stiffness being greater than or equal to a specified value is also incorporated. Although results are presented for relations of a form $I(x) = \rho A^n(x)$ where ρ and n are constants it will be shown how similar problems with ρ being a function of x or with relations of the form attempted by Brach can be very similarly treated for obtaining approximate numerical solutions.

FORMULATION OF THE PROBLEM

Given an Euler–Bernoulli beam of a specified length and volume (mass) resting on a continuous elastic foundation and allowed to vibrate freely under the influence of any given varying axial load and/or dead mass distribution and under various boundary conditions (mixed or not—with or without elastic restraints), the problem is to determine the distribution of material along the length of the beam so as to maximize the first mode frequency (design objective) subject to the constraint that the minimum stiffness (area) of the beam is not less than a specified value A_0 (inequality constraint), see Fig. 1.

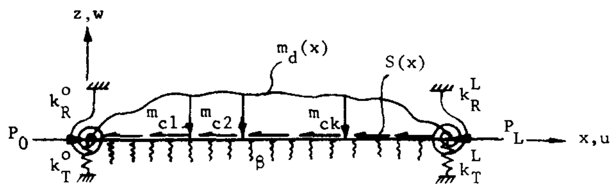

FIG. 1. A typical beam on a continuous elastic foundation with elastically restrained ends under arbitrarily varying axial load and dead mass distribution.

The problem of the optimal vibrating beam is thus a max–min problem, i.e. requiring simultaneously that the lowest eigenvalue (first mode frequency) be maximized with respect to variations in stiffness subject to the equality constraint that the total mass (volume or weight) is a constant. This last requirement leads to the optimality condition.

The two variational problems are thus posed simultaneously in order to find the minimum eigenvalue and the best distribution of material such that the lowest eigenvalue is a maximum.

In the following development consideration will be restricted to those beams for which the cross-sectional moment of inertia and area are related by a relation of the form

$$I(x) = \rho A^n(x) \tag{1}$$

ρ being a shape constant. The index n can assume all positive values but results will be presented for only three specific values of n namely $n = 1, 2$ and 3.

Rayleigh's principle states that in a natural mode of vibration of a conservative system the frequency of vibration is stationary. Furthermore, at the first mode ω_1^2 (the square of the frequency) is a minimum, that is to say, ω^2 corresponding to any kinematically admissible mode shape $w(x)$, will be higher than the exact first mode frequency. Ignoring the effect of shear deformation the Rayleigh quotient is given by

$$\omega_1^2 = \min \frac{\int_0^L EI(x)[w''(x)]^2\,dx + U_s - \int_0^L q(x)[w'(x)]^2\,dx}{\int_0^L m(x)[w(x)]^2\,dx + \int_0^L m_d(x)[w(x)]^2\,dx + \sum_{i=1}^k m_{ci} w_i^2} \tag{2}$$

where

$$U_s = \text{twice the energy of the spring supports}$$

$$= k_T^0 w^2|_0 + k_T^L w^2|_L + k_R^0 w'^2|_0 + k_R^L w'^2|_L + \beta \int_0^L w^2 \, dx$$

β being the foundation modulus

$$q(x) = P_0 - \int_0^x S(\xi)\, d\xi$$

such that

$$\int_0^L S(\xi)\, d\xi = P_0 - P_L \qquad \text{(see Fig. 1)}$$

$m(x)$ = the mass per unit length and is related to $A(x)$ by the relation $m(x) = (\gamma/g)A(x)$; γ being the specific weight of the material and g the acceleration due to gravity.

Since $I(x) = \rho A^n(x)$ equation (2) can be written as

$$\omega_1^2 = \min \frac{\int_0^L E\rho A^n(x)[w''(x)]^2 \, dx + U_s - \int_0^L q(x)[w'(x)]^2 \, dx}{\int_0^L (\gamma/g)A(x)[w(x)]^2 \, dx + \int_0^L m_d(x)[w(x)]^2 \, dx + \sum_{i=1}^k m_{ci} w_i^2}.$$

The necessary condition for ω_1^2 to be a minimum w.r.t. w is that $\delta_{w,w'}(\omega_1^2) = 0$. This leads to the governing equation of motion of the beam and the associated boundary conditions. These are:

$$[E\rho A^n w'']'' - \omega_1^2 \left(\frac{\gamma}{g} A + m_d\right) w + \beta w + [q(x) w']' = 0 \tag{3}$$

$$\left.\begin{aligned} E\rho A^n w'' - k_R^0 w' &= 0 \\ (E\rho A^n w'')' + k_T^0 w + q w' &= 0 \end{aligned}\right\} \text{at } x = 0 \tag{4}$$

$$\left.\begin{aligned} E\rho A^n w'' + k_R^L w' &= 0 \\ (E\rho A^n w'')' - k_T^L w + q w' &= 0 \end{aligned}\right\} \text{at } x = L. \tag{5}$$

In addition, there are the conditions of continuity of displacement, slope, moment and known discontinuities of shear at the points of application of the concentrated dead masses, if any.

Next, it is required to maximize ω_1^2 with respect to variations in $A(x)$ subject to

$$\int_0^L A \, dx = V. \tag{6}$$

Hence, the new functional that must be extremized is

$$(\omega_1^2)^* = \frac{\int_0^L E\rho A^n(x)[w''(x)]^2 \, dx + U_s - \int_0^L q(x) w'^2 \, dx}{\int_0^L (\gamma/g)A(x)[w(x)]^2 \, dx + \int_0^L m_d(x)[w(x)]^2 \, dx + \sum_{i=1}^k m_{ci} w_i^2} - \lambda_1 \left[\int_0^L A \, dx - V\right]$$

where λ_1 is an undetermined Lagrange multiplier. The necessary condition for $(\omega_1^2)^*$ to be stationary with respect to variations in $A(x)$ is

$$\int_0^L \left\{E\rho n A^{n-1} w''^2 - \lambda_1 \left[\int_0^L \left(\frac{\gamma}{g} A + m_d\right) w^2 \, dx + \sum_{i=1}^k m_{ci} w_i^2\right] - \omega_1^2 \frac{\gamma}{g} w^2\right\} \delta A \, dx = 0.$$

Hence, if δA is arbitrary, i.e. the area is not prescribed then the above implies that

$$E\rho n A^{n-1} w''^2 - \omega_1^2 \frac{\gamma}{g} w^2 = c = \text{constant}. \tag{7}$$

Equation (7) is valid only in those regions where the area is not prescribed. In other regions, in the event that the area as determined by the use of equation (7) happens to be less than A_0, the constraint $A = A_0$ has to be satisfied.

Multiplication of equation (7) throughout by A followed by integration from $x = 0$ to $x = L$ yields

$$n \int_0^L E\rho A^n w''^2 \,\mathrm{d}x - \omega_1^2 \int_0^L \frac{\gamma}{g} A w^2 \,\mathrm{d}x = c \int_0^L A \,\mathrm{d}x$$

or

$$c = \frac{2(nU_B - U_T)}{V}.$$

Equation (7) can therefore be written as

$$E\rho n A^{n-1} w''^2 - \omega_1^2 \frac{\gamma}{g} w^2 = \frac{2}{V}(nU_B - U_T). \tag{8a}$$

Notice that in the case of a beam with classical boundary conditions for $n = 1$, the constant c is zero while for the same beam with elastic restraints the constant c is negative. It can also be seen that equations (3) through (7) remain unchanged in the event that ρ is a function of x. For relations of the form

$$I(x) = c_0 + \rho A(x)$$

although equations (3)–(5) have to be modified, as it will be seen later, the corresponding matrix equations in terms of finite elements remain the same in form while the optimality condition for this case is nothing more than what one would obtain for $n = 1$, i.e.

$$E\rho w''^2 - \omega_1^2 \frac{\gamma}{g} w^2 = c_3 = \text{constant}. \tag{8b}$$

METHOD OF SOLUTION OF THE PROBLEM

The proposed method is the finite element displacement method which reduces to the Rayleigh–Ritz method when the assumed displacement function satisfies compatibility exactly. Further, as will be seen later, the optimality condition, equation (8a), when transformed in terms of finite elements is much simpler to handle.

The details of the finite element displacement method as applied to vibrating beams can be found in several references (Refs. [17] and [18]) and hence are not reproduced here.

In terms of finite elements the equation of motion together with the boundary conditions becomes

$$[[K] - \omega_1^2 [M]]\{q\} = \{0\} \tag{9}$$

where $[K]$ is the assembled nonsingular stiffness matrix for the entire beam including the effect of the applied axial loading, ω_1^2 is the lowest eigenvalue, $[M]$ is the assembled mass matrix for the entire beam including the effect of dead mass if any and $\{q\}$ is the vector of unrestrained degrees of freedom of the beam. Having determined ω_1^2 and the corresponding eigenvector $\{q\}_1$ by the solution of the eigenvalue problem as specified by equation (9) the strain energy and the kinetic energy densities in each element can be determined as

$$\frac{U_{bi}}{v_i} = \frac{\frac{1}{2}\{q_i\}^T[k_i]\{q_i\}}{A_i l_i} \qquad i = 1, 2 \ldots m \tag{10a}$$

and

$$\frac{U_{ti}}{v_i} = \frac{\frac{1}{2}\{q_i\}^T[M_i]\{q_i\}}{A_i l_i} \qquad i = 1, 2 \ldots m. \tag{10b}$$

Next, the optimality condition is transformed in terms of the finite elements. Multiplying equation (8a) throughout by A and integrating over the extent of the ith element one obtains

$$\int_{x_i}^{x_{i+1}} E\rho n A^n w''^2 \,\mathrm{d}x - \omega_1^2 \int_{x_i}^{x_{i+1}} \frac{\gamma}{g} A w^2 \,\mathrm{d}x = c \int_{x_i}^{x_{i+1}} A \,\mathrm{d}x$$

i.e.

$$2nU_{bi} - 2U_{ti} = cv_i$$

or

$$n\left(\frac{U_{bi}}{v_i}\right) - \left(\frac{U_{ti}}{v_i}\right) = \frac{c}{2} = c_1. \tag{11}$$

Equation (11) can be written as

$$\frac{n(U_{bi}/v_i)}{c_1 + (U_{ti}/v_i)} = 1 \quad \text{if } c_1 > 0 \tag{12a}$$

or

$$\frac{n(U_{bi}/v_i) - c_1}{(U_{ti}/v_i)} = 1 \quad \text{if } c_1 < 0. \tag{12b}$$

Equation (11) is the necessary optimality condition in terms of finite elements.

Note that for $I(x) = \rho A^n(x)$ the constant c_1 is given by

$$c_1 = (nU_B - U_T)/V$$

while for $I(x) = c_0 + \rho A(x)$ the constant c_1 is given by

$$c_1 = \left[-\frac{1}{V}\int_0^L Ec_0 w''^2 \,\mathrm{d}x\right].$$

Unconstrained optimization

The objective of this optimization is to make the ratio $n(U_{bi}/v_i)/(c_1 + (U_{ti}/v_i))$ equal to unity. This is similar to the objective in Ref. [5] where it was required to make $(U_i V/v_i U)$ equal to unity. Hence a similar procedure is employed.

One starts with a uniform beam, i.e. a beam having a uniform cross-section complying with the given volume V. Then using equation (10) the strain energy and the kinetic energy densities in each of the elements can be determined. These distributions of strain and kinetic energy densities are used for deciding the inertias of the elements for the next iteration.

Assume the rth iteration begins with the ith finite element having the moment of inertia I_i^r $(i = 1, 2 \ldots m)$. After determining the associated eigenvalue and eigenvector, the average strain and kinetic energy densities for each element and the average strain and kinetic energy densities for the whole beam can be computed; these quantities are denoted by U_{bi}^r/v_i^r, U_{ti}^r/v_i^r $(i = 1 \ldots m)$ and U_B^r/V and U_T^r/V where

$$U_B^r = \sum_{i=1}^{m} U_{bi}^r, \qquad U_T^r = \sum_{i=1}^{m} U_{ti}^r$$

and

$$V = \sum_{i=1}^{m} v_i^r = \text{specified volume.}$$

The inertias for the next iteration are assumed to be given by the following relation

$$I_i^{r+1} = C^{r+1} I_i^r \left[\frac{n(U_{bi}^r/v_i^r)}{[c_1^r + (U_{ti}^r/v_i^r)]} \right]^p \quad \text{if } c_1^r > 0$$

or

$$I_i^{r+1} = C^{r+1} I_i^r \left[\frac{[n(U_{bi}^r/v_i^r) - c_1^r]}{(U_{ti}^r/v_i^r)} \right]^p \quad \text{if } c_1^r < 0$$

where the exponent p is assumed to be positive and the constant C^{r+1} is determined from the constant volume constraint. Next, it can be shown that as long as the ratio inside the brackets is different from unity a value of $p > 0$ exists which guarantees that $(\omega_1^2)^{r+1} \geq (\omega_1^2)^r$ (see Refs. [5] and [19]).

The initial value of p can be assumed to be 1 or less and the iterative scheme can be continued with the value of p for as long as $(\omega_1^2)^{r+1} \geq (\omega_1^2)^r$. If it so happens that at some stage $(\omega_1^2)^{r+1} < (\omega_1^2)^r$ then the value of p is reduced by a factor of half or a quarter and the iteration is repeated. This process is continued until no substantial change either in the value of ω_1^2 or the moment of inertia distribution is possible and the optimality condition is essentially satisfied. The initial value of p assumed and the successive factors used to scale it down during the process decide the number of iterations required for complete convergence. The iterative procedure thus guarantees a monotonic convergence to the maximum first mode frequency though not always via a monotonic convergence of the energy density ratio.

The existence of p to guarantee $(\omega_1^2)^{r+1} \geq (\omega_1^2)^r$ can also be shown when $I(x) = c_0 + \rho A(x)$, the volume constraint that needs to be satisfied in this case being

$$\int_0^L \frac{I^{r+1}}{\rho} \mathrm{d}x = V_1 = V - \frac{c_0}{\rho} L.$$

Constrained optimization

In the case of the inequality constraint, assume the value of the prescribed minimum inertia to be I_0.

The constrained optimization proceeds exactly in the same manner as the unconstrained optimization till such point at which the inertias of some elements violate the inequality constraint. The inertias of those elements are arbitrarily set equal to the prescribed minimum value I_0 while the inertias of the remaining elements have to be recalculated. Assume the volume of the j elements with prescribed inertias to be V_c. The new value of c_1 can be calculated for the remaining $(m-j)$ elements. Let this new value of c_1 be denoted by c_2. Hence the new inertias of these $(m-j)$ elements are given by

$$I_i^{r+1} = C^{r+1}\frac{n(U_{bi}/v_i)^r}{c_2^r+(U_{ti}/v_i)^r} \quad \text{if } c_2^r > 0$$

where C^{r+1} is to be determined from the constraint

$$\sum_{i=1}^{(m-j)}\left(\frac{I_i^{r+1}}{\rho}\right)^{1/n} l_i = (V-V_c)$$

the summation in the above equation extends only over those $(m-j)$ elements which do not violate the inequality constraint. It should be noted that in the case of inequality constraint the quantity

$$\left(n\frac{U_{bi}}{v_i}-\frac{U_{ti}}{v_i}\right)$$

will be equal to a constant only over those $(m-j)$ elements which do not violate the inequality constraint. For the j elements with prescribed inertias the aforementioned quantity will have different values.

RESULTS AND CONCLUSIONS

The criterion used for specifying convergence is

$$\left[\left(\frac{nU_{bi}-U_{ti}}{v_i}\right)_{\max}\Bigg/\left(\frac{nU_{bj}-U_{tj}}{v_j}\right)_{\min}-1{\cdot}0\right]\times 100 \le 0{\cdot}50.$$

(a) A uniform cross-section freely vibrating simply-supported with $I = \rho A$ satisfies the optimality condition trivially, i.e.

$$\frac{U_{bi}-U_{ti}}{v_i} = \frac{U_B-U_T}{V} = 0, \qquad i = 1\ldots m$$

and hence no increase in the first mode frequency is possible. On the other hand for a freely vibrating simply-supported beam with $I(x) = \rho A^n(x)$, $n = 2$ and 3 a finite (6 per cent for $n = 2$ and 11·5 per cent for $n = 3$) increase of the first mode frequency is obtained (see Fig. 2). This 6 per cent increase for $n = 2$ compares very favorably with the 6·6 per cent increase obtained by Niordson. It can be seen that in Fig. 2 only ten elements need be considered because of symmetry and complete convergence can be obtained within a matter of about five to six iterations.

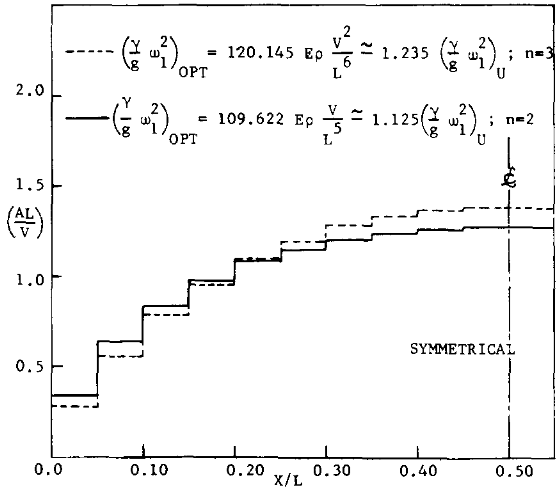

FIG. 2. Optimum area distribution for a beam with $k_T^0 = \infty, k_R^0 = 0, k_T^L = \infty, k_R^L = 0$; (i) $n = 2$, (ii) $n = 3$.

(b) Figure 3 shows a simply-supported beam on a continuous elastic foundation of moderate stiffness and the corresponding increases of the first mode frequency for $m = 10$ and $m = 20$ respectively, m being the number of elements.

(c) Figure 4 shows the effect of an axial tension load on the optimum first mode frequency of a simply-supported beam with $I(x) = \rho A^2(x)$.

(d) In the absence of the dead mass distribution and/or a compressive axial load ($P/P_{cr} \simeq 0{\cdot}25$ or higher) and/or an inequality constraint except for the simply-supported beam, all beams with other boundary conditions do not seem to possess a finite optimum first mode frequency. This does not seem to be in agreement with the conclusions of Ref. [16] for the case of a cantilever with no dead mass at the tip. For this case Ref. [16] obtains

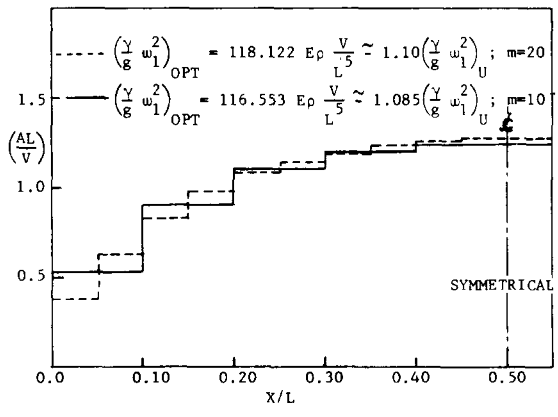

FIG. 3. Optimum area distribution for a beam with $k_T^0 = \infty, k_R^0 = 0, k_T^L = \infty, k_R^L = 0$; $\beta = 10\ E\rho V^2/L^6$; (i) $m = 10$, (ii) $m = 20$.

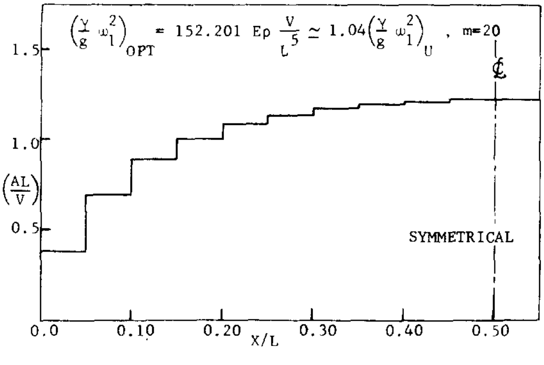

FIG. 4. Optimum area distribution for a beam with $k_T^0 = \infty$, $k_R^0 = 0$, $k_T^L = \infty$, $k_R^L = 0$; $P_0 = P_L = -5\,E\rho V^2/L^4$; $n = 2$.

finite optimum frequencies for $n = 2$ and 3. With the finite element representation of a beam, which is in essence a stepped beam, it seems that frequencies highly in excess of those reported in Ref. [16] can be obtained with increasing number of elements used to model the beam.

(e) In Figs. 5, 7, 8 and 10, η denotes the ratio of the total dead mass to the total structural mass of the beam. Figure 5 shows a cantilever beam with $I(x) = \rho A^2(x)$ vibrating with a concentrated dead mass at the tip. Results are presented for this with $m = 10$ and $m = 20$. For the assumed ratio of η the results show a good correlation with those of Ref. [16].

(f) Figure 6 shows the effect of an inequality constraint on the optimum area distribution and the first mode frequency of a freely vibrating cantilever beam with $I(x) = \rho A^2(x)$. Results are presented for this beam with $m = 10$ and $m = 20$.

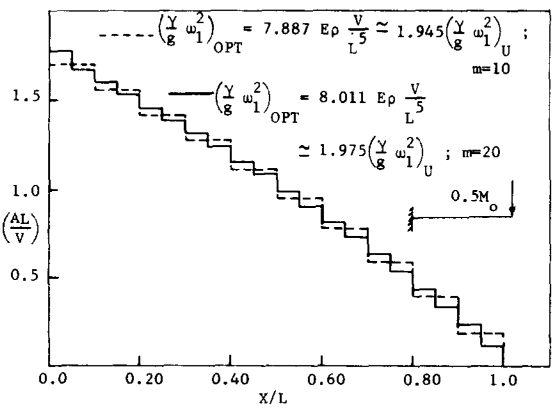

FIG. 5. Optimum area distribution for a beam with $k_T^0 = \infty$, $k_R^0 = \infty$, $k_T^L = 0$, $k_R^L = 0$; $\eta = 0{\cdot}50$, $x_c = L$; $n = 2$; (i) $m = 10$, (ii) $m = 20$.

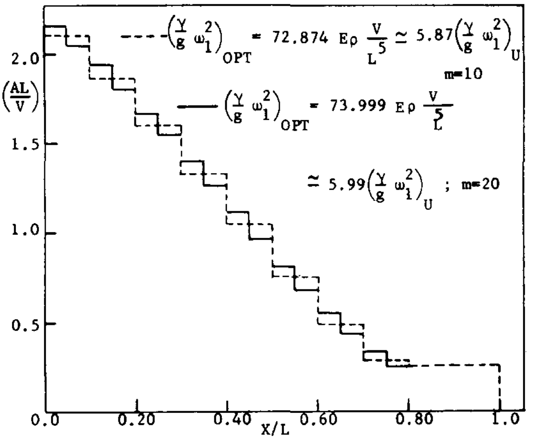

FIG. 6. Optimum area distribution for a beam with $k_T^0 = \infty$, $k_R^0 = \infty$, $k_T^L = 0$, $k_R^L = 0$; $A \geq 0{\cdot}2725\ V/L$; $n = 2$; (i) $m = 10$, (ii) $m = 20$.

(g) Figure 7 shows a cantilever beam with $I(x) = \rho A^2(x)$ vibrating under the combined influence of a constant compressive axial load and a linearly varying dead mass distribution.

(h) Figure 8 shows a clamped–clamped beam vibrating under the influence of a concentrated dead mass at the center.

(i) Figure 9 shows a clamped–clamped beam vibrating under the influence of a constant compressive axial load for $m = 20$ and $m = 40$ with $I(x) = \rho A^2(x)$. No finite frequency seems to exist for this beam under the influence of a tension load.

(j) Figure 10 shows two typical cases of a clamped–pinned beam; firstly the effect of an inequality constraint on the beam with $I(x) = \rho A(x)$ and secondly the effect of a uniformly distributed dead mass on the beam with $I(x) = \rho A^2(x)$.

(k) Figures 11–13 show three typical cases of the elastically restrained beams vibrating under the influence of a uniformly distributed dead mass.

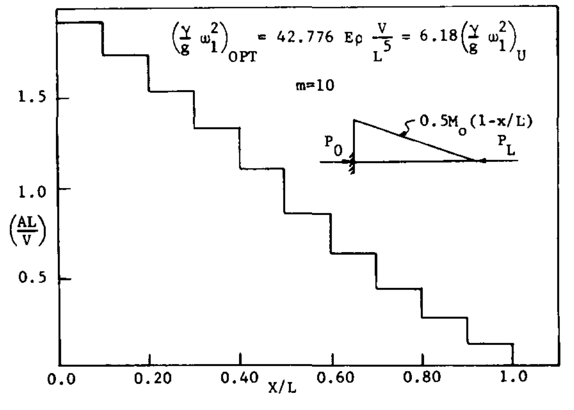

FIG. 7. Optimum area distribution for a beam with $k_T^0 = \infty$, $k_R^0 = \infty$, $k_T^L = 0$, $k_R^L = 0$; $\eta = 0{\cdot}25$; $P_0 = P_L = E\rho V^2/L^4$; $n = 2$.

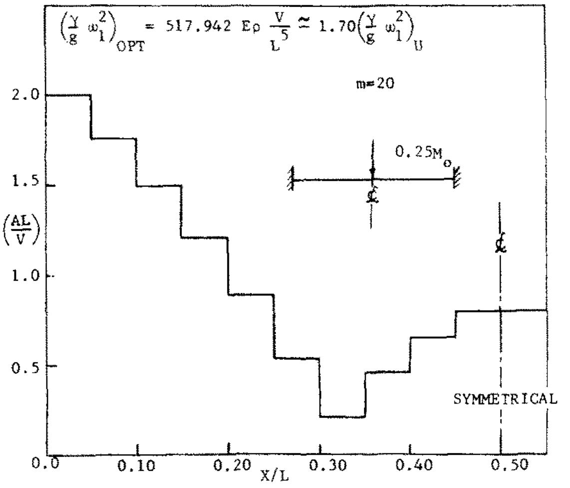

FIG. 8. Optimum area distribution for a beam with $k_T^0 = \infty$, $k_R^0 = \infty$, $k_T^L = \infty$, $k_R^L = \infty$; $\eta = 0{\cdot}25$, $x_c = 0{\cdot}5\,L$; $n = 2$.

(l) In most of the cases 99 per cent of the optimum frequency is obtained in a matter of a few iterations provided the corresponding continuous system does possess a finite frequency.

(m) It is worthwhile noting that the effect of shear deformation and rotary inertia can be very easily accounted for without changing the basic form of the optimality condition. nU_{bi} would then be replaced by $(nU_{bi} + U_{si})$ where U_{si} is the strain energy of the ith element due to the effect of shear deformation and U_{ti} would correspond to the total kinetic energy of the ith element which is composed of the kinetic energy of translation and n times the kinetic energy of rotation of the beam. The element stiffness and mass matrices would have

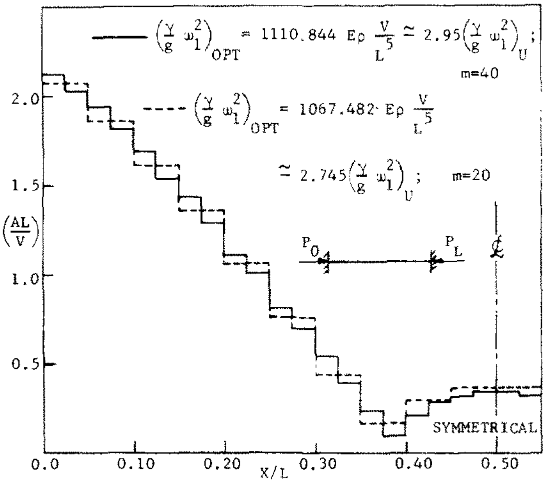

FIG. 9. Optimum area distribution for a beam with $k_T^0 = \infty$, $k_R^0 = \infty$, $k_T^L = \infty$, $k_R^L = \infty$; $P_0 = P_L = 10\,E\rho V^2/L^4$; $n = 2$; (i) $m = 20$, (ii) $m = 40$.

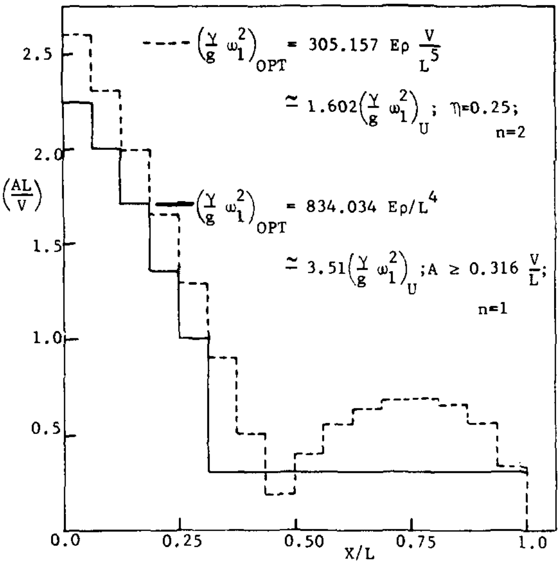

FIG. 10. Optimum area distribution for a beam with $k_T^0 = \infty$, $k_R^0 = \infty$, $k_T^L = \infty$, $k_R^L = 0$; $m = 16$; (i) $A \geq 0{\cdot}316\ V/L$, $n = 1$; (ii) $m_d(x) = 0{\cdot}25\ M_0/L$, $n = 2$.

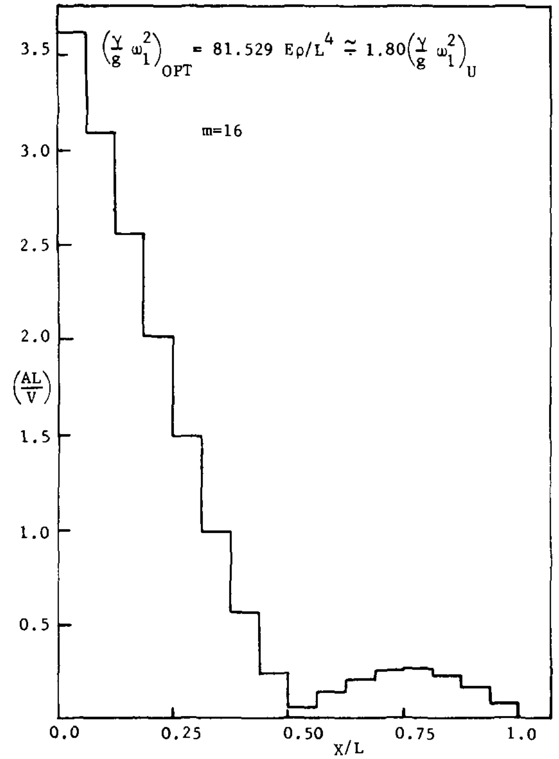

FIG. 11. Optimum area distribution for a beam with $k_T^0 = \infty$, $k_R^0 = \infty$, $k_T^L = 25\ E\rho V/L^4$, $k_R^L = 0$; $m_d(x) = M_0/L$; $n = 1$.

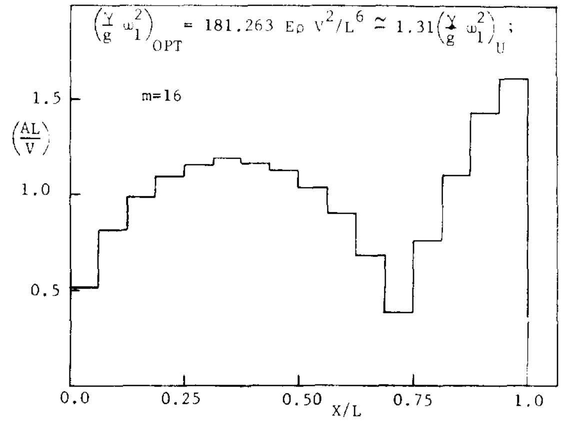

FIG. 12. Optimum area distribution for a beam with $k_T^0 = \infty$, $k_R^0 = 0$, $k_T^L = \infty$, $k_R^L = 25\,E\rho V^3/L^4$; $m_d(x) = M_0/2L$; $n = 3$.

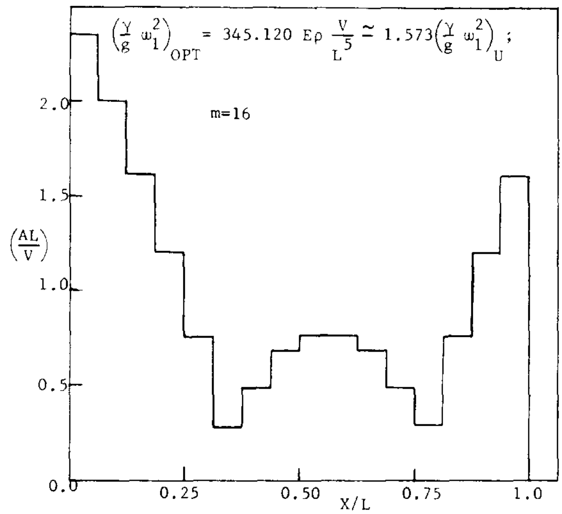

FIG. 13. Optimum area distribution for a beam with $k_T^0 = \infty$, $k_R^0 = \infty$, $k_T^L = \infty$, $k_R^L = 25\,E\rho V^2/L^3$; $m_d(x) = M_0/L$; $n = 2$.

to be altered to take this effect into account and hopefully the same optimization procedure can be used to converge to the optimum first mode frequency. Including these effects would then perhaps ensure a finite frequency for the beam regardless of the boundary conditions and in the absence of the dead mass distribution and/or a compressive axial load and/or an inequality constraint.

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Абстракт—Используется формулировка перемещения конечного элемента, с целью доведения до максимума собственной частоты первого вида колебаний колебающейся балки заданного объема, с упруго заделанными концами и лежащей на непрерывном упругом основании, подверженной ограничению минимума допускаемого момента инерции. Для круглого поперечного сечения, с моментом ниерции и площадью, определенных формулой $I = \rho A^{\eta}$ (ρ и η положительные постоянные) условие постоянности сводится к зависимости между плотностями энергии деформации и кинетической энергии. Начиная с однородной балкой, применяется итеративный процесс с целью определения сходимости к оптимуму распределения материала и максимальной частоты первого вида колебаний. Даются результаты для разных краевых условий, с эффектом и без эффекта распределения некоторой заданной, произвольно изменяющейся осебой нагрубки и, либо, или распределения собственной массы, для $\eta = 1, 2$ и 3.